2 Simple Algebra (see Jones *et al*. (2000) pages 201-205)

Algebra can be regarded as generalised arithmetic, replacing numbers (1, 8, 29, etc.) by **letters** or **symbols** (x, y, α, δ, etc.) as variables and allowing us to make general statements about the relationships between them as formulae or in the form of equations or inequalities which have to be solved.

An **algebraic expression** is any mathematical formula which contains both **numbers (coefficients)** and **letters (terms)**, where each letter just stands for a number whose value you don't want to specify in advance.

Example: {32 + (4*1)}/6 is an **arithmetic expression**, which you can **evaluate** to give the answer 6.

(32 + 4y)/6 is an algebraic expression, in which y can take **any numerical value** you like. You can only evaluate the expression if you specify a value for y. For example, if you specify y = 1, then the expression becomes (32 + 4)/6 as before, and can be evaluated to give the answer 6.

It is useful to simplify algebraic expressions; the following examples illustrate the rules.

- Where the terms are similar (i.e. contain the same symbol or variable) they are collected by adding and subtracting the coefficients. Consider the expression 14p –3p + 5p. Now 14 – 3 + 5 = 16, so the expression can be simplified to 16p.
- Where the terms are not similar they cannot be combined by simply adding or subtracting the coefficients. The expression 12p – 5q + 4r cannot be simplified further. However, partial simplification is sometimes possible. In the expression 12y – 3z + 5z – 8y, we can treat the terms in y and the terms in z separately to give 4y + 2z.
- Different powers of a variable cannot be treated as like terms. The expression $3y + 5y^2$ cannot be simplified since the terms in y and y^2 are not considered as like terms.
- Multiplication and division follow the rules for manipulating numbers. For example,

$$a * b = ab = ba = b * a$$

$$3y * 5z = 3 * 5 * y * z = 15yz$$

$$a * (-b) = -ab; \quad (-a) * b = -ab; \quad (-a) * (-b) = ab$$

$$\frac{a}{-b} = -\frac{a}{b} = \frac{-a}{b}; \qquad \frac{-a}{-b} = \frac{a}{b}$$

- **Powers**: Just as 4^3 means $4*4*4$, we can write y^3 as a shorthand for $y*y*y$.The rules for powers of numbers can be generalised and expressed compactly using algebraic notation.

$a^0 = 1$ i.e. any number raised to the power 0 equals 1.

$a^1 = a$ i.e. any number raised to the power 1 equals the number.

$a^m * a^n = a^{m+n}$ i.e. when multiplying two powers of the same number we add the indices.

$a^m \div a^n = \frac{a^m}{a^n} = a^{m-n}$ i.e. when dividing two powers of the same number we subtract the indices.

$(a^m)^n = a^{mn}$ i.e. the power of a power is achieved by multiplying the indices.

$a^{-n} = \frac{1}{a^n}$.

$(ab)^n = a^n b^n$.

2.1 Examples

$5^2 * 5^3 = 5^5$**;**

$$4^5 \div 4^3 = \frac{4*4*4*4*4}{4*4*4} = 4^{5-3} = 4^2 = 16$$

$$(3^4)^2 = 3^4 * 3^4 = 3^8 = 3^{4*2}$$

$$3^{-2} = \frac{1}{3^2} = \frac{1}{9}$$

2.2 Fractional powers

The square root of y can be written as $\sqrt{y}$ or $y^{1/2}$ and, in general the nth root of y is written $\sqrt[n]{y} = y^{1/n}$.

Example

$\sqrt{49} =$ square root of 49 = 7, since 7 * 7 = $7^2 = 49$

2.3 Expansion of single brackets

To expand 3(2y + 4) means multiplying the number outside the bracket with each term inside.

3(2y + 4) = 3*2*y + 3*4 = 6y + 12.

2.4 Expansion of two brackets

Expanding an expression means multiplying it out.
To expand (2y + 4)(y – 3) each term in the first bracket is multiplied by each term in the second bracket. To make sure that nothing is missed out, it is sensible to follow the same order every time.

(2y + 4)(y – 3) = $2y^2 - 6y + 4y - 12 = 2y^2 - 2y - 12$

2.5 Algebraic Manipulation of Equations

The need to arrange formulae is particularly important when it comes to making one of the variables in an algebraic expression or formula into the subject of that formula. For example, A is the **subject** of the formula

$$A = \pi r^2$$

where A is the area of a circle with radius r.

The basic rule in rearranging algebraic equations is very easily stated:

DO THE SAME TO BOTH SIDES

that is,

- add the same thing to both sides
- subtract the same thing from both sides
- multiply both sides by the same thing
- divide both sides by the same thing

Example 2.1

Make r the subject in $A = \pi r^2$ and find the radius of a circle with area 10 m^2, where A = area, $r = \text{radius}$ and $\pi = 3.142$

Solution

$$A = \pi r^2$$

divide both sides by π: $\qquad r^2 = \dfrac{A}{\pi}$

take square roots: $\qquad r = \sqrt{\dfrac{A}{\pi}}$

when A =10 m^2, the radius (r) is given by

$$r = \sqrt{\frac{A}{\pi}} = \sqrt{\frac{10\,m^2}{3.142}} = \sqrt{3.183\,m^2} = 1.784\,m$$

Example 2.2

Darcy's Law [see ENV 104 (Hydrological processes) and ENV 211 (Catchment hydrology)] underpins most calculations of groundwater flow. It can be written as

$$Q = A\,K_s\left(\frac{dH}{L}\right)$$

where

Q = groundwater discharge

A = the cross-sectional area (cm^2) of soil/rock through which the water is flowing

K_s = the soil or rock's saturated hydraulic conductivity (cm/hr)

dH = the change in total potential or total head (cm)

L = length over which the change in head is measured (cm)

Make K_s the subject and find K_s if Q = 34920 $cm^3\ hr^{-1}$; A = 44.18 cm^2; L =10 cm; dH =3.9 cm.

Solution

$$Q = A\,K_s\left(\frac{dH}{L}\right)$$

divide both sides by A: $$\frac{Q}{A} = K_s\left(\frac{dH}{L}\right)$$

multiply both sides by L: $$\frac{LQ}{A} = K_s(dH)$$

divide both sides by dH: $$K_s = \left(\frac{Q}{A}\right)\left(\frac{L}{dH}\right)$$

Substituting the numerical values for Q, A, L and dH in above equation

$$K_s = \left(\frac{34920\ \text{cm}^3\text{hr}^{-1}}{44.18\ \text{cm}^2}\right)\left(\frac{10\ \text{cm}}{3.9\,\text{cm}}\right) = 2026.67\ \text{cm}\,\text{hr}^{-1}$$

Example 2.3 [see ENV 104 (Hydrological processes) and ENV 211 (Catchment hydrology)]
River discharge (Q) is simply a product of the mean velocity (V) and the cross-sectional area (A) of the river, that is

$$Q = V * A$$

(a) If V = 0.22 m/s and A = 0.0822 m^2. Find Q and what are the units?

Solution

$$Q = V * A = 0.22\ \text{m}\ \text{s}^{-1} * 0.0822\ \text{m}^2 = 0.0181\,\text{m}^3\,\text{s}^{-1}$$

(b) In the above example, make V the subject.

$$Q = V * A$$

divide both sides by A: $V = \dfrac{Q}{A}$

Example 2.4: Dilution gauging: Constant injection method [see ENV 104 (Hydrological processes) and ENV 211 (Catchment hydrology)]

This method is based on the two component mixing equation, i.e.

$(Q+q)C_2$	=	qC_1	+	QC_0
total downstream flow * mixed concentration	=	tracer input rate * tracer concentration	+	upstream flow * background concentration

where Q is the unknown upstream river discharge, q is the tracer discharge, C_2 is the mixed downstream concentration, C_1 is the concentration of the tracer to be added, and C_0 is the background tracer concentration in the river (may be zero). Make Q the subject.

Solution

expand $(Q+q)C_2$: $QC_2 + qC_2 = qC_1 + QC_0$

subtract QC_0 from both sides: $QC_2 + qC_2 - QC_0 = qC_1$

subtract qC_2 from both sides: $QC_2 - QC_0 = qC_1 - qC_2$

i.e. $Q(C_2 - C_0) = q(C_1 - C_2)$

divide both sides by $(C_2 - C_0)$: $Q = \dfrac{q(C_1 - C_2)}{(C_2 - C_0)}$

Example 2.5 [see ENV 104 (Hydrological processes) and ENV 211 (Catchment hydrology)]

The data below relate to a dilution gauging using the constant rate injection method. Estimate the stream discharge in m^3s^{-1}.

Tracer input rate (q): $5\ m\ell\ s^{-1}$

Tracer Concentration (C_1): $200\ g\ \ell^{-1}$

Background tracer concentration in the river (C_0): $6\ \mu g\ \ell^{-1}$

Mixed downstream concentration (C_2): $86\ \mu g\ \ell^{-1}$

Solution

$$Q = \frac{q(C_1 - C_2)}{(C_2 - C_0)}$$

note that $200\,g\ \ell^{-1} = 200*10^6\ \mu g\,\ell^{-1}$

$$Q = \frac{5\,m\ell\,s^{-1}(200*10^6 - 86)\mu g\,\ell^{-1}}{(86-6)\mu g\,\ell^{-1}} = \frac{5(200*10^6 - 86)}{(86-6)}\,m\ell\,s^{-1}$$

$$Q = \frac{5(200*10^6 - 86)}{(86-6)}\,m\ell\,s^{-1} = \frac{5(199999914)}{80}\,m\ell\,s^{-1}$$

note that 1 litre = $10^{-3}\ m^3$; $\therefore 1\,m\ell = 10^{-3}\,\ell = 10^{-6}\,m^3$

$$\therefore \quad Q = \frac{5(200*10^6 - 86)}{(86-6)} \, m\ell s^{-1} = \frac{5(199999914)}{80} \, m\ell s^{-1}$$

$$= \frac{5*199999914*10^{-6}}{80} \, m^3 s^{-1} = 12.5 \, m^3 s^{-1}$$

Example 2.6 [see ENV 123, ENCH 203 and ENCH 204: see Jones *et al*. (2000) page 31]

If specific conditions are defined by pressures P_1 and P_2 with corresponding volumes V_1 and V_2 and temperatures T_1 and T_2 for a given mass of gas, then

$$\frac{P_1 V_1}{T_1} = \frac{P_2 V_2}{T_2}$$

This is useful when estimating the change in volume of a mass of air when atmospheric temperature and pressure are changed.

(a) Make V_2 the subject and (b) A gas has a volume of 1 litre (ℓ) at 283 K (10^0C). What will be the final volume if the temperature is increased to 293 K (20^0C) and pressure is constant (i.e. $P_1 = P_2$).

Solution

(a) $$\frac{P_2 V_2}{T_2} = \frac{P_1 V_1}{T_1}$$

Divide both sides by P_2: $$\frac{V_2}{T_2} = \frac{P_1 V_1}{T_1 P_2}$$

Multiply both sides by T_2: $$V_2 = \frac{P_1 V_1 T_2}{T_1 P_2}$$

Since $P_1 = P_2$ $$V_2 = \frac{V_1 T_2}{T_1}$$

(b) $$V_2 = \frac{V_1 T_2}{T_1} = \frac{1\ell * 293\,K}{283\,K} = 1.035\,\ell$$

Example 2.7 [see ENV 104 (Hydrological processes) and ENV 211 (Catchment hydrology)]

Catchment water balance equation can be written as

$$P = E + Q \pm \Delta S$$

where P is the precipitation, Q is the river discharge, E is the evapo-transpiration (i.e., the combined losses of open-water evaporation, wet-canopy evaporation / interception loss, and transpiration) and ΔS is the change in subsurface water storage. Rearrange above equation to make E the subject.

Solution

$$P = E + Q \pm \Delta S$$

- Q from both sides: $P - Q = E + Q \pm \Delta S - Q = E \pm \Delta S$

$\pm \Delta S$ to both sides: $E = P - Q \pm \Delta S$

Note that over a period of a year or more, the change in subsurface storage (ΔS) averages to zero, and can therefore be omitted from the calculations.

Example 2.8 [see ENV 105(Atmosphere, weather and climate)]

The gravitational acceleration, g, is related to the Earth's mass, M, by the equation

$$g = \frac{GM}{r^2}$$

where G is a known physical constant and r is the Earth's radius.

(a) Rearrange above equation into an expression for M
(b) Estimate the Earth's mass, M if $g = 9.81\,m\,s^{-2}$, r = 6370 km and $G = 6.672 * 10^{-11}\,m^3 kg^{-1} s^{-2}$

Solution

(a)

$$g = \frac{GM}{r^2}$$

$$\therefore\ GM = g r^2$$

$$\therefore\ M = \frac{g r^2}{G}$$

(b) Since both g and G are given in units which include meters, convert Earth's radius r into metres, i.e. r = 6370 km = $6.37 * 10^6$ m

$$M = \frac{g\,r^2}{G} = \frac{9.81*(6.37*10^6)^2}{6.672*10^{-11}} = \frac{9.81*6.37*6.37*10^{12}}{6.672*10^{-11}} = \frac{9.81*6.37*6.37*10^{12}*10^{11}}{6.672}$$

$$\therefore\ M = \frac{(9.81*6.37*6.37)*10^{23}}{6.672} = 59.66*10^{23}\ \text{kg}$$

Example 2.9 [see ENV 105(Atmosphere, weather and climate)]

The volume of the Earth can be estimated using the standard formula for the volume, V of a sphere of radius r. This is

$$V = \frac{4\pi r^3}{3}$$

The density ρ (see Table 2.1 for a list of Greek letters) is related to mass and volume by

$$\text{Density, } \rho = \frac{\text{mass}}{\text{volume}} = \frac{M}{V}$$

Show that the average density of the Earth is given by $\rho = \dfrac{3g}{4G\pi r}$

Solution

$$\rho = \frac{\text{mass}}{\text{volume}} = \frac{M}{V}$$

but from example 2.8, $M = \dfrac{g\,r^2}{G}$

$$\therefore \rho = \frac{M}{V} = M \div V = \frac{g\,r^2}{G} \div \frac{4\pi r^3}{3} = \frac{g\,r^2}{G} * \frac{3}{4\pi r^3} = \frac{3g}{4G\pi r}$$

Example 2.10 [see ENV 105(Atmosphere, weather and climate)]

Find the density of the Earth, if $g = 9.81\,\text{m}\,\text{s}^{-2}$, r = 6370 km , $\pi = 3.142$ and $G = 6.672*10^{-11}\ \text{m}^3\text{kg}^{-1}\text{s}^{-2}$

Solution

Note that r = 6370 km = $6.37*10^6$ m

From above

$$\rho = \frac{3g}{4G\pi r} = \frac{3*9.81}{4*6.672*10^{-11}*3.142*6.37*10^{6}}$$

$$\therefore \rho = \frac{3*9.81}{4*6.672*3.142*6.37*10^{-5}} = \frac{3*9.81*10^{5}}{4*6.672*3.142*6.37} = 0.0551*10^{5} = 5510\,\text{kg}\,\text{m}^{-3}$$

and this is more than five times the density of water (which is around 1000 $\text{kg}\,\text{m}^{-3}$).

2.6 The Greek alphabet

Table 2.1 Upper case and lower case letters of the Greek alphabet.

Greek letters (upper case)	Greek letters (lower case)	Name
Α	α	alpha
Β	β	beta
Γ	γ	gamma
Δ	δ	delta
Ε	ε	epsilon
Ζ	ζ	zeta
Η	η	eta
Θ	θ	theta
Ι	ι	iota
Κ	κ	kappa
Λ	λ	lambda
Μ	μ	mu
Ν	ν	nu
Ξ	ξ	xi
Ο	ο	omicron
Π	π	pi
Ρ	ρ	rho
Σ	σ	sigma
Τ	τ	tau
Υ	υ	upsilon
Φ	φ	phi
Χ	χ	chi
Ψ	ψ	psi
Ω	ω	omega